VII. ORTHOGONAL DECOMPOSITION OF $L^2(\mathbb{R})$ INDUCED BY MRA

In this chapter and next, we will look into a general Multiresolution Analysis and discuss some of its important properties. First let us recall the definition of a Multiresolution Analysis.

Definition 1. *For any $n \in \mathbb{Z}$, let V_n be a subspace of $L^2(\mathbb{R})$. Suppose $\{V_n\}_{n\in\mathbb{Z}}$ satisfies the following conditions:*

(a)For any $n \in \mathbb{Z}$, $V_n \subset V_{n+1}$;

(b)For any $n \in \mathbb{Z}$, $f(x) \in V_n \Longleftrightarrow f(2x) \in V_{n+1}$;

(c)$\overline{\cup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R})$;

(d)$\cap_{n\in\mathbb{Z}} V_n = \{0\}$;

(e)There exists $\varphi(x) \in V_0$ such that $\{\varphi(x-l)|l \in \mathbb{Z}\}$ is a complete orthonormal system of V_0.

Then $\{V_n\}_{n\in\mathbb{Z}}$ is called a **Multiresolution Analysis***.*

The function $\varphi(x) \in V_0$ is called a scalling function for the Multiresolution Analysis $\{V_n\}_{n\in\mathbb{Z}}$. Just as in the specific case of Haar Multiresolution analysis, the orthogonal complement of V_0 in V_1 is well defined, we call it W_0. Namely, $W_0 = V_1 \ominus V_0$. Equivalently, $V_1 = V_0 \oplus W_0$. In general, since for any $n \in \mathbb{Z}$, we have $V_n \subset V_{n+1}$, the orthogonal complement of V_n in V_{n+1} is well defined and we call it W_n. Namely, $W_n = V_{n+1} \ominus V_n$. Equivalently, $V_{n+1} = V_n \oplus W_n$.

First let us note that most of the general discussions in Chapter 6 applies in this general situation. In particular, we can transplant Lemma 3 of Chapter 6 to this general situation. The original proof applies in this general situation so it is omitted.

Lemma 1. *Let $\{V_n\}_{n\in\mathbb{Z}}$ be a Multiresolution Analysis and $W_n = V_{n+1} \ominus V_n$ for each $n \in \mathbb{Z}$. Then for any $n \in \mathbb{Z}$,*

$$f(x) \in W_n \Longleftrightarrow f(2x) \in W_{n+1}.$$

We also quote Lemma 4 of Chapter 6 below verbatum.

Lemma 2. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be subspaces of $L^2(\mathbb{R})$. If $f(x) \in \mathcal{K}_1 \iff f(2x) \in \mathcal{K}_2$, then the following are equivalent:*

(a) $\{f_n(x)\}_{n\in\mathbb{Z}}$ is a complete orthonormal system of $\mathcal{K}_1$.

(b)$\{\sqrt{2}f_n(2x)\}_{n\in\mathbb{Z}}$ is a complete orthonormal system of $\mathcal{K}_2$.

Now, if for a general Multiresolution Analysis, we can find a complete orthonormal system of W_0, as we have done in Chapter 6 for the Haar multiresolution case, then we can decompose and reconstruct functions in $L^2(\mathbb{R})$ in the same fashion as we have done with the aid of Haar multiresolution analysis. In particular, Theorem 1 of Chaper 6 is still valid in principal.

Theorem 1. *Let $\{V_n\}_{n\in\mathbb{Z}}$ be a Multiresolution Analysis and $W_n = V_{n+1} \ominus V_n$ for each $n \in \mathbb{Z}$. Let φ be the scaling function of $\{V_n\}_{n\in\mathbb{Z}}$ and ψ be a function in W_0 such that $\{\psi(x-l)|l \in \mathbb{Z}\}$ is a complete orthonormal system of W_0. Let $\psi_{j,l}(x) = 2^{\frac{j}{2}}\psi(2^j x - l)$ and $\varphi_{j,l}(x) = 2^{\frac{j}{2}}\varphi(2^j x - l)$ for each $j, l \in \mathbb{Z}$. Then for any $h \in L^2(\mathbb{R})$ and for each $j \in \mathbb{Z}$,*

$$P_{V_{j+1}}h = P_{V_j}h + P_{W_j}h.$$

Specifically,

$$\sum_{l\in\mathbb{Z}}\langle h, \varphi_{j+1,l}\rangle\varphi_{j+1,l} = \sum_{l\in\mathbb{Z}}\langle h, \varphi_{j,l}\rangle\varphi_{j,l} + \sum_{l\in\mathbb{Z}}\langle h, \psi_{j,l}\rangle\psi_{j,l}.$$

In the next Chapter, we will be able to show that for a general Multiresolution Analysis, we can indeed find a complete orthonormal system of W_0. Thus, once we have a Multiresolution analysis, we will always be able to do decomposition and reconstruction of functions in $L^2(\mathbb{R})$, in a similar way to the Haar case. The specific formulas of decomposition and reconstruction will be different for each individual multiresolution analysis. It depends on the specific relation between the scaling function φ and the function $\psi \in W_0$.

There is another generalization we want to make in this Chapter. Recall that the Haar function $H(x)$ has been proved to be an orthogonal wavelet in Chapter 4, it is also proved the $\{H(x-l)|l \in \mathbb{Z}\}$ is a complete orthonormal system of W_0 of Haar multiresolution analysis. In the rest of this chapter, we want to show that for a general multiresolution analysis, if $\psi \in L^2(\mathbb{R})$ is such a function that $\{\psi(x-l)|l \in \mathbb{Z}\}$ is a complete orthonormal system of W_0 induced by that general multiresolution analysis, then ψ is a wavelet in n$L^2(\mathbb{R})$.

We will do this through proving a decomposition Theorem below.

Theorem 2. *Let $\{V_n\}_{n\in\mathbb{Z}}$ be a Multiresolution Analysis and $W_n = V_{n+1} \ominus V_n$ for each $n \in \mathbb{Z}$. Then for any $f \in L^2(\mathbb{R})$, there is a unique sequence of functions $\{f_k\}_{k\in\mathbb{Z}}$ such that $f_k \in W_k$ for each $k \in \mathbb{Z}$, f_k's are mutually orthogonal,*

$$f = \lim_{m,n\to+\infty} \sum_{k=-m}^{k=n} f_k$$

where the convergence is in $L^2(\mathbb{R})$ norm and

$$||f||_2^2 = \lim_{m,n\to+\infty} \sum_{k=-m}^{k=n} ||f_k||^2.$$

Remark Without causing any confusion, we can also write

$$f = \lim_{m,n\to+\infty} \sum_{k=-m}^{k=n} f_k$$

in the above theorem as $f = \sum_{k\in\mathbb{Z}} f_k$ with the understanding that the convergence is in $L^2(\mathbb{R})$ norm and write

$$||f||_2^2 = \lim_{m,n\to+\infty} \sum_{k=-m}^{k=n} ||f_k||^2$$

as $||f||_2^2 = \sum_{k\in\mathbb{Z}} ||f_k||^2$. We delay the proof of Theorem 2 and first see one of its implications.

Theorem 3. *Let $\{V_n\}_{n\in\mathbb{Z}}$ be a Multiresolution Analysis and $W_n = V_{n+1} \ominus V_n$ for each $n \in \mathbb{Z}$. Let $\psi \in L^2(\mathbb{R})$ be such a function that $\{\psi(x-l)|l \in \mathbb{Z}\}$ is a complete orthonormal system of W_0. Then ψ is a orthogonal wavelet in $L^2(\mathbb{R})$.*

Proof. Note that according to Lemma 1, in the case of a general multiresolution analysis, for any $n \in \mathbb{Z}$,

$$f(x) \in W_n \Longleftrightarrow f(2x) \in W_{n+1}.$$

Since $\psi \in L^2(\mathbb{R})$ is a function that $\{\psi(x-l)|l \in \mathbb{Z}\}$ is a complete orthonormal system of W_0, by applying Lemma 2, we see that $\{2^{\frac{k}{2}}\psi(2^k x - l)|l \in \mathbb{Z}\}$ is a complete orthonormal system of W_k for each $k \in \mathbb{Z}$. So for any function $f_k \in W_k$, by Lemma 4 of Chapter 1, we have

$$||f_k||_2^2 = \sum_{l\in\mathbb{Z}} |\langle f_k, 2^{\frac{k}{2}}\psi(2^k x - l)\rangle|^2.$$

Thus for any $f \in L^2(\mathbb{R})$, according to Theorem 3, we have

$$||f||_2^2 = \sum_{k\in\mathbb{Z}} ||f_k||^2 = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} |\langle f_k, 2^{\frac{k}{2}}\psi(2^k x - l)\rangle|^2.$$

Since according to Theorem 3, $f = \sum_{n\in\mathbb{Z}} f_n$ with $f_n \in W_n$ and W_n's are mutually orthogonal. So for any fixed $k \in \mathbb{Z}$, because $2^{\frac{k}{2}}\psi(2^k x - l) \in W_k$ for all $l \in \mathbb{Z}$, so for all $l \in \mathbb{Z}$, $2^{\frac{k}{2}}\psi(2^k x - l)$ is orthogonal to each W_n with $n \neq k$. Thus

$$\langle f, 2^{\frac{k}{2}}\psi(2^k x - l)\rangle = \langle \sum_{n\in\mathbb{Z}} f_n, 2^{\frac{k}{2}}\psi(2^k x - l)\rangle = \langle f_k, 2^{\frac{k}{2}}\psi(2^k x - l)\rangle$$

for all $l \in \mathbb{Z}$. Thus

$$||f||_2^2 = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} |\langle f_k, 2^{\frac{k}{2}}\psi(2^k x - l)\rangle|^2 = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} |\langle f, 2^{\frac{k}{2}}\psi(2^k x - l)\rangle|^2.$$

Again by Lemma 4 of Chapter 1, $\{2^{\frac{k}{2}}\psi(2^k x - l) | k, l \in \mathbb{Z}\}$ is an complete orthonormal system of $L^2(\mathbb{R})$, thus ψ is an orthogonal wavelet in $L^2(\mathbb{R})$ by definition. $\square$

Lastly, let us prove Theorem 2 through several steps. First we consider the orthogonal decomposition of V_0.

Lemma 3. *Let $\{V_n\}_{n\in\mathbb{Z}}$ be a Multiresolution Analysis and $W_n = V_{n+1} \ominus V_n$ for each $n \in \mathbb{Z}$. Then for any $f \in V_0$, there is a unique sequence of functions $\{f_{-k}\}_{k\in\mathbb{N}}$ such that $f_{-k} \in W_{-k}$ for each $k \in \mathbb{N}$, f_{-k}'s are mutually orthogonal,*

$$f = \lim_{n\to+\infty} \sum_{k=1}^{k=n} f_{-k}$$

where the convergence is in $L^2(\mathbb{R})$ norm.

Proof. We first prove the existence of such f_{-k}'s. To this end, we let $f_{-k} = P_{W_{-k}} f$ for all $k \in \mathbb{N}$ and let $g_n = f - \sum_{k=1}^{n} f_{-k}$ for each $n \in \mathbb{N}$. We are going to prove the sequence $\{g_n\}_{n=1}^{\infty}$ converges to 0 under the $L^2(\mathbb{R})$ norm.

First, we are going to prove that $\{g_n\}_{n=1}^{\infty}$ is convergent. To reach this goal, we only need to prove that $\{g_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R})$. Note that $\sum_{k=1}^{n} f_{-k} \in W_{-1} \oplus W_{-2} \oplus ... \oplus W_{-n}$, and $g_n \in V_{-n}$, so they are orthogonal to each other, hence by

$$||f||_2^2 = ||g_n + \sum_{k=1}^{n} f_{-k}||_2^2 = ||g_n||_2^2 + ||\sum_{k=1}^{n} f_{-k}||_2^2 \geq ||\sum_{k=1}^{n} f_{-k}||_2^2 = \sum_{k=1}^{n} ||f_{-k}||_2^2.$$

Namely

$$\sum_{k=1}^{n} ||f_{-k}||_2^2 \leq ||f||_2^2$$

holds for any $n \in \mathbb{N}$. This means that

$$\sum_{k=1}^{\infty} ||f_{-k}||_2^2 \leq \infty.$$

So for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that for any $m > n > N$, we have

$$\sum_{k=n+1}^{m} ||f_{-k}||_2^2 \leq \varepsilon^2.$$

namely,

$$||g_m - g_n||_2^2 = ||\sum_{k=n+1}^{m} f_{-k}||_2^2 = \sum_{k=n+1}^{m} ||f_{-k}||_2^2 \leq \varepsilon^2.$$

This means $\{g_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R})$ and thus converges to some $g \in L^2(\mathbb{R})$. In view of $g_n = f - \sum_{k=1}^{n} f_{-k}$, we conclude that $\sum_{k=1}^{n} f_{-k}$ also converges, and we denote

$$g = f - \sum_{k=1}^{\infty} f_{-k}.$$

Now we are going to prove that $g = 0$. Indeed, since for any $N \geq 1$, the sequence $\{g_n\}_{n=N}^{\infty} \subset V_{-N}$ and $\{g_n\}_{n=N}^{\infty}$ also converges in $L^2(\mathbb{R})$ norm to g, so $g \in V_N$. Hence

$$g \in \bigcap_{N=1}^{\infty} V_{-N} = \bigcap_{n=-1}^{-\infty} V_n = \bigcap_{n\in\mathbb{Z}} V_n = \{0\}$$

Thus $g = 0$ and we get

$$f = \lim_{n\to+\infty} \sum_{k=1}^{k=n} f_{-k} = \sum_{k\in\mathbb{N}} f_{-k}$$

where the convergence is in $L^2(\mathbb{R})$ norm.

Lastly, the uniqueness part of the proof is left to the reader. □

Next we consider the orthogonal decomposition of $L^2(\mathbb{R}) \ominus V_0$.

Lemma 4. *Let $\{V_n\}_{n\in\mathbb{Z}}$ be a Multiresolution Analysis and $W_n = V_{n+1} \ominus V_n$ for each $n \in \mathbb{Z}$. Then for any $f \in L^2(\mathbb{R}) \ominus V_0$, there is a unique sequence of functions $\{f_k\}_{k=0}^{\infty}$ such that $f_k \in W_k$ for each $k \in \mathbb{N} \cup \{0\}$, f_k's are mutually orthogonal,*

$$f = \lim_{n\to+\infty} \sum_{k=0}^{k=n} f_k$$

where the convergence is in $L^2(\mathbb{R})$ norm.

Proof. First we prove the existence of such f_k's. To this end , let $f_k = P_{W_k} f$ for all $k \in \mathbb{N} \cup \{0\}$. Since

$$\overline{\bigcap_{n=1}^{\infty} V_n} = L^2(\mathbb{R})$$

and $V_n \subset V_{n+1}$ for any $n \in \mathbb{Z}$, hence for any $f \in L^2(\mathbb{R}) \ominus V_0 \subset L^2(\mathbb{R})$, and any $\varepsilon > 0$, there is an natural number $N \in \mathbb{N}$ and a function $g_N \in V_N$ such that $||f - g_N||_2 < \varepsilon$.

For this fixed N, since

$$P_{V_N} f = \sum_{k=0}^{N-1} f_k \in V_N,$$

according to the properties of orthogonal projection, we have

$$||f - \sum_{k=0}^{N-1} f_k||_2 = ||f - P_{V_N} f||_2 \leq ||f - g_N||_2 < \varepsilon.$$

Now for any integer $n > N - 1$, since $f - \sum_{k=0}^{n} f_k \in L^2(\mathbb{R}) \ominus V_{n+1}$, while $\sum_{k=N}^{n} f_k \in V_{n+1}$, so functions $f - \sum_{k=0}^{n} f_k$ and $\sum_{k=N}^{n} f_k$ are orthogonal to each other, hence by P theorem,

$$||\sum_{k=0}^{N-1} f_k||_2^2 = ||f - \sum_{k=0}^{n} f_k + \sum_{k=N}^{n} f_k||_2^2 = ||f - \sum_{k=0}^{n} f_k||_2^2 + ||\sum_{k=N}^{n} f_k||_2^2$$

So for the above $\varepsilon > 0$, and above fixed N, and any integer $n > N - 1$, we see that

$$||f - \sum_{k=0}^{n} f_k||_2 \leq ||f - \sum_{k=0}^{N-1} f_k||_2 < \varepsilon$$

which means $f = \sum_{k=0}^{\infty} f_k$.

The uniqueness part is left to the reader. □

Now we are ready to prove Theorem 2.

Proof of Theorem 2. First we want to prove that for any $\varepsilon > 0$, there are natural numbers M, N, such that for all integers $n > N$ and $m > M$, we have

$$||f - \sum_{k=-n}^{m} f_k||_2 < \varepsilon.$$

Since for any $f \in L^2(\mathbb{R})$, there is a unique $g \in V_0$ and a unique $h \in L^2(\mathbb{R}) \ominus V_0$ such that $f = g + h$ and $f \perp h$. Now we can apply Lemma 3 and Lemma 4 to functions g and h respectively to get the result.

Secondly, we want to prove that $||f||_2^2 = \lim_{m,n\to+\infty} \sum_{k=-m}^{k=n} ||f_k||^2$. This is the consequence of what we have already proved and the following lemma. The details of the application is left to the reader. $\square$

Lemma 5. *Suppose that $\mathcal{H}$ is a Hilbert space, $f \in \mathcal{H}$, $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H}$. If f_k's are mutually orthogonal and $\sum_{k=1}^{\infty} f_k = f$, then $\sum_{k=1}^{\infty} ||f_k||^2 = ||f||^2$.*

Proof. We want to prove that for any $\varepsilon > 0$, there is a natural number N such that for any integer $n > N$, we have $\big|||f||^2 - \sum_{k=1}^{n} ||f_k||^2\big| < \varepsilon$.

Let us assume for the moment that $f \neq 0$. For any $\varepsilon > 0$, we take $\varepsilon_1 = \min\{\frac{\varepsilon}{3||f||}, ||f||\}$, thus $\varepsilon_1 > 0$. Since $\sum_{k=1}^{\infty} f_k = f$, so there is an natural number N, such that for any integer $n > N$, we have

$$\big|||f|| - ||\sum_{k=1}^{n} f_k||\big| < ||f - \sum_{k=1}^{n} f_k||\varepsilon_1 \leq ||f||$$

hence

$$||\sum_{k=1}^{n} f_k|| \leq \big|||f|| - ||\sum_{k=1}^{n} f_k||\big| + ||f|| \leq 2||f||.$$

Therefore, for any integer $n > N$,

$$\big|||f||^2 - \sum_{k=1}^{n} ||f_k||^2\big| = \big|||f||^2 - ||\sum_{k=1}^{n} f_k||^2\big| = \big|||f|| - ||\sum_{k=1}^{n} f_k||\big| \cdot \big(||f|| + ||\sum_{k=1}^{n} f_k||\big)$$

$$< \varepsilon_1 \cdot 3||f|| \leq \frac{\varepsilon}{3||f||} \cdot 3||f|| = \varepsilon.$$

The case when $f = 0$ is left to the reader. $\square$